

Scattering amplitudes at strong coupling for $4K$ gluons

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Abstract

In this paper we study the scattering amplitudes at strong coupling for the case where the number of gluons is a multiple of four. This is an important missing piece in [30]. The tricky point for $n = 4K$ is that there is some accidental degeneracy in such case. We explain this point in detail and show that a non-trivial monodromy around infinity was developed by the world-sheet coordinate transformation appearing in the computation. It turns out that besides solving the Y system, we also need to calculate T functions to compute the full amplitudes. We show that the T functions can be derived by taking a limit of Y functions of a higher-point case. As a check, we obtain the known result of eight-point in AdS_3 in [28].

1 Introduction

Scattering amplitudes are central quantities in quantum field theory. The knowledge of their behavior at higher loops and at strong coupling may be instrumental in understanding the problems such as quark confinement or quantum gravity. While it is very hard to do such calculations in QCD or in gravitational theories, many significant developments in past several years have shown that it may be possible to have a non-perturbative understanding of S -matrix in $\mathcal{N} = 4$ SYM.

Based on the explicit perturbative calculation, Bern, Dixon and Smirnov proposed a non-perturbative conjecture for planar MHV amplitudes in $\mathcal{N} = 4$ SYM, for all number of gluons up to all loops [1]. This is now well-known as BDS ansatz. The idea was also indicated before in [2]. This ansatz was supported by the later calculation of two-loop five-point amplitude and four-point amplitude up to five loops [3, 4]. The (generalized) unitarity method plays an essential role for doing the higher-loop calculation [5, 6].

At strong coupling, by using AdS/CFT duality [7], a recipe for calculating scattering amplitudes was also proposed by Alday and Maldacena [8]. The problem is reduced to calculating the area of minimal surfaces in AdS_5 ending on a null polygon at the boundary, where the shape of the polygon is determined by the momenta of external gluons. Due to the similar prescription for Wilson loop [9, 10], this indicated that there may be a duality between amplitudes and Wilson loops at weak coupling, which was soon proved to be true at one loop for general n points, and for four and five points at two loops [11, 12, 13].

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At the same time, the BDS ansatz was also questioned by the study of amplitudes at strong coupling for large number of gluons [14]. Later the explicit weak coupling two-loop six-point calculations showed that the BDS ansatz is incorrect while the duality between amplitudes and Wilson loops is still true [15, 16]. On the other hand, the BDS ansatz gives the correct conformal anomaly of Wilson loops [13]¹. Therefore, under the assumption of the amplitude/Wilson loop duality, the difference between BDS ansatz and the true result should be a (dual) conformal invariant quantity, which is usually referred to as the “remainder function”. To fully understand planar MHV amplitudes, the main problem is to understand this mysterious remainder function². A numerical program for calculating two-loop Wilson loop was developed in [23], and some properties of the remainder functions beyond six-point were studied in [24]. The analytic calculation of remainder function for six-point was also performed in [25, 26].

Unlike at weak coupling, the calculation of amplitudes at strong coupling is a geometric minimal surface problem. For the simplest four-point case [8], the solution of the minimal surface was obtained by some guess, or by doing conformal transformations to a cusp solution [27]. But it is very hard to find solutions for higher-point cases. Remarkably, in a series of papers [28, 29, 30], Alday, Maldacena and collaborators developed a method which makes it possible to calculating the area of minimal surface with general null polygonal boundary conditions, where the integrability of the system plays an essential role [31, 32, 33]. Using this method, one can calculate the area directly without the need of constructing the explicit solution of the minimal surface. Let us briefly mention some key steps here.

The first important trick is the Pohlmeyer reduction [34, 35, 36] (see also [37, 38, 39, 40] for some recent developments). By using this reduction, solving the classical string equations and the Virasoro constraints becomes solving a Hitchin system (with a Z_4 projection). A very important fact for Hitchin system is that the equations can be promoted by introducing a spectral parameter ζ . This turns out to be instrumental for solving the problem. In particular, by introducing this auxiliary parameter, the cross ratios can be promoted to a function of spectral parameter. The functional relations between cross ratios can be organized in a framework of the so called Y system [41, 42] (see also [43, 44]), where Y functions are the cross ratios. Under this framework, one can write a set of integral equations, where the boundary conditions can be very nicely embedded via WKB approximation at large and small ζ [45, 46], and finally, the non-trivial part of the area can be expressed as the free energy of the Y system.

While the above prescription works well for the case where the number of gluons is odd, it can not be applied directly to the case where the number of gluons n is even³. For such cases, one may obtain the result by taking a limit of $(n+1)$ -point case. This is relatively trivial when $n=4K+2$ [30]. However, the calculation is much more subtle when $n=4K$, i.e. the number of gluons is a multiple of four. As we will show, such cases are special in that a world-sheet coordinate transformation appearing in the computation develops a non-trivial monodromy around infinity. This makes the calculation of the so called cutoff part and periods part much more non-trivial. In the simple AdS_3 case, a prescription was given in [28], but a full prescription for AdS_5 case is still unknown. This is the problem that we consider in this paper.

¹At one loop level, the conformal anomaly has been proved for general n -point amplitudes [17].

²We should mention that there are other very important problems about understanding non-MHV (and also non-planar) amplitudes. The dual conformal supersymmetry [18], fermionic T-duality [19, 20], and Grassmannian integral [21, 22] are some important developments along these lines.

³This is because the intersection form of the Riemann surface appearing in the calculation is only invertible for the case where n is odd [30].

We will provide a general prescription for the computation of the cutoff part. In $n \neq 4K$ cases, the cutoff part is trivial and can be uniquely written in terms of only adjacent kinematic invariants. But in the cases of $n=4K$, these are the main complications. We will show the problem can be solved by introducing two extra equations which involve non-adjacent kinematic invariants. These equations also involve the so called T -functions. The parts that depend on the T -functions are defined as extra part, while the remaining parts are defined as BDS-like part. We show that the T functions can be calculated as a limit of Y functions. Our prescription reproduces the known AdS_3 result, which provides a strong check for the consistency and validity of the method.

The paper is organized as follows. In section 2 we review the general structure of amplitudes at strong coupling. In section 3 we study in detail the origin of the subtlety in $n=4K$ case and calculate the cutoff part for such case. In section 4, we calculate T functions as a limit of Y functions. In section 5 we make a conjecture for the periods part. We present the explicit eight-point result in section 6. Section 7 contains some discussions. We give a brief summary for the Y system in the appendix.

2 Structure of amplitudes at strong coupling

The general structure of amplitudes at strong coupling can be given as

$$A = A_{\text{div}} + A_{\text{BDS-like}} + A_{\text{extra}} + A_{\text{periods}} + A_{\text{free}} . \quad (1)$$

The free and periods parts are basically the parts that can be calculated via Y system [30]. The cutoff part is constituted of A_{div} , $A_{\text{BDS-like}}$ and A_{extra} parts. As we will see in next section, the extra part appears only in the $n=4K$ case. We emphasize that although the free and periods parts may be the most non-trivial part of the amplitudes, the cutoff part also contains important physical information. For example, for four and five-point cases in particular, the cutoff part gives the whole result, therefore contains the whole physics.

Let us look at the origin of each part more closely.

By Pohlmeyer reduction, the area of the minimal surface can be written as

$$A = 2 \int d^2 z \text{Tr}[\Phi_z \Phi_{\bar{z}}] , \quad (2)$$

where Φ is a component of flat connection of the corresponding Hitchin system. The boundary conditions of the problem require that $\text{Tr}[\Phi_z \Phi_{\bar{z}}] \rightarrow (P(z)\bar{P}(\bar{z}))^{1/4}$ for large z , therefore one can regularize the area by subtracting the asymptotic divergent part as

$$A_{\text{free}} = 2 \int d^2 z \text{Tr}[\Phi_z \Phi_{\bar{z}}] - 2 \int d^2 w , \quad dw = P(z)^{1/4} dz , \quad (3)$$

where for n -point $P(z)$ is a polynomial of degree $n-4$. This part is called free part since it turns out to be the free energy of the corresponding Y system [30]⁴

$$A_{\text{free}} = \sum_s \frac{m_s}{2\pi} \int_{-\infty}^{+\infty} d\theta \cosh \theta \log \left[(1 + Y_{1,s})(1 + Y_{3,s})(1 + Y_{2,s})^{\sqrt{2}} \right] . \quad (4)$$

⁴Notice that in [30] the method used to deriving A_{free} is only valid when n is odd, since the intersection form of the Riemann surface is singular in other cases. One can argue that by starting from odd n result and taking a large m limit, we have the same expression of free part for all other cases.

The area can then be written as

$$A = A_{\text{free}} + 2 \int d^2 w . \quad (5)$$

The second term is still divergent, therefore needs regularization. It is convenient to introduce a cutoff surface Σ and consider the integral $\int_{\Sigma} d^2 w$ ⁵. Notice the surface Σ still contains complicated branch cut information which is given by the polynomial $P(z)$. To simplify the problem, we can introduce another surface Σ_0 with the same cutoff but with simpler internal structure. Then we can separate the second term further into two parts

$$2 \int_{\Sigma} d^2 w = A_{\text{periods}} + A_{\text{cutoff}} , \quad (6)$$

where

$$A_{\text{periods}} = 2 \int_{\Sigma} d^2 w - 2 \int_{\Sigma_0} d^2 w , \quad A_{\text{cutoff}} = 2 \int_{\Sigma_0} d^2 w . \quad (7)$$

While $n \neq 4K$ we can define Σ_0 corresponding to a polynomial whose zeros are all degenerate at the origin. Then the periods part can be defined explicitly as

$$A_{\text{periods}} = 2 \int d^2 z ([P(z)\bar{P}(\bar{z})]^{1/4} - |z|^{n/2-2}) , \quad (8)$$

which can be expressed in terms of periods around cycles of the Riemann surface⁶, therefore explains why it's called periods part. Using Y system, this part can be calculated together with the free part when n is odd. For the case of $n=4K+2$, the result can also be obtained by taking a large mass limit of $n=4K+3$ case [30]. However the $n = 4K$ case is more tricky, we will discuss this part in section 5.

The remaining part is the cutoff part. If the number of gluons is not a multiple of four, the calculation is very simple. There is no extra part in such cases. And besides the universal divergent part, the BDS-like part turns out to be the unique solution of the dual conformal Ward identity which is expressed in terms of only adjacent kinematic invariants $x_{i,i+2}^2$. Explicitly, for $n = 4K + 2$, we have

$$A_{\text{BDS-like}} = -\frac{1}{8} \sum_{i=1}^n \left(\ell_i^2 + \sum_{k=0}^{2K} \ell_i \ell_{i+1+2k} (-1)^{k+1} \right) , \quad (9)$$

while $n = 4K + 1, 4K + 3$ we have

$$A_{\text{BDS-like}} = -\frac{1}{4} \sum_{i=1}^n \left(\ell_i^2 + \sum_{k=0}^{2K} \ell_i \ell_{i+1+2k} (-1)^{k+1} \right) , \quad (10)$$

where

$$\ell_i \equiv \log x_{i,i+2}^2 . \quad (11)$$

However, when $n=4K$, the calculation of cutoff part becomes much more complicated, due to the existence of a monodromy around infinity which we will discuss in detail in next section.

⁵In the weak coupling calculation, dimensional regularization is more convenient. At strong coupling as a geometrical problem the cutoff regularization appears to be very natural. This cutoff regularization may be related to the off-shell and Higgs regularization at weak coupling [47, 48, 49].

⁶The Riemann surface appearing here is defined as algebraic curves which is related to the polynomial $P(z)$. For AdS_5 case, the Riemann surface is defined as $x^4=P(z)$ which is a quadruple branch cover of Riemann sphere. While for AdS_3 case, it is only a double branch cover defined by $x^2=p(z)$ (where $P(z) = p(z)^2$ in such case).

3 Cutoff part

We calculate the cutoff part in this section. We first review the embedding coordinate and cutoff regulator. Then we discuss why the calculation is tricky for the $n=4K$ case from various points of view. We show that one can calculate the cutoff part for such case by introducing two new equations involving non-adjacent kinematic invariants, and also T functions which give the extra part.

3.1 Embedding coordinates and cutoff regulator

It is convenient to work in the embedding coordinates of AdS_5 space

$$X \cdot X \equiv -X^+ X^- + X^\mu X_\mu = -1, \quad \mu = 0, 1, 2, 3, \quad (12)$$

where

$$X^\pm = X^{-1} \pm X^4. \quad (13)$$

The boundary of AdS_5 space can be defined as $\hat{X} = X/R$, by taking $R \rightarrow \infty$

$$\hat{X}^2 = -\hat{X}^+ \hat{X}^- + \hat{X}^\mu \hat{X}_\mu = 0, \quad \hat{X} \sim \lambda \hat{X}. \quad (14)$$

The relation between embedding coordinates and Poincaré coordinates is

$$x_\mu = \frac{X_\mu}{X_+}, \quad r = \frac{1}{X^+} = \frac{1}{\hat{X}^+ R}, \quad (15)$$

where in Poincaré coordinates the boundary is defined at $r \rightarrow 0$, which is consistent with taking $R \rightarrow \infty$.

To impose the cutoff, we need to understand the asymptotic behavior of the minimal surface. An important trick to impose the boundary condition is to change the world-sheet coordinate from original z coordinate to w coordinate, via $dw = P(z)^{1/4} dz$ [29]. In the new w coordinate, every cover of w -plane contains only four cusps⁷, and the minimal surface with n cusps covers the w -plane $n/4$ times. Due to the non-trivial polygonal boundary condition, the solution of the minimal surface has different asymptotic behaviors near different cusps, which can be described by the so call ‘‘Stokes phenomenon’’ [45, 46]. Each cusp corresponds to one Stokes sector, and each stokes sector has one smallest solutions s_i that decay fastest to the boundary. Therefore, for every cover of w plane, we have four Stokes sectors and four smallest solutions.

Now we can regularize the surface. As in the usual way, we introduce a cutoff for the radius of AdS_5

$$r > \mu, \quad \text{or equivalently} \quad X^+ < \frac{1}{\mu}. \quad (16)$$

The asymptotic behavior of the solution near each cusp can be given in w -plane as⁸

$$X_i^A \simeq \hat{X}_i^A \times \{e^{2\text{Re}[w]}, e^{2\text{Im}[w]}, e^{-2\text{Re}[w]}, e^{-2\text{Im}[w]}\}. \quad (17)$$

⁷Since every cover of w -plane contains four cusp, the asymptotic behavior should be the same as the four-cusp solution which is well understood. This is why it's easy to impose boundary condition by using w coordinate.

⁸Notice that we have rotated the original w -plane by $\pi/4$ to simplify the exponential function. They are actually similar to the u, v -plane in [28].

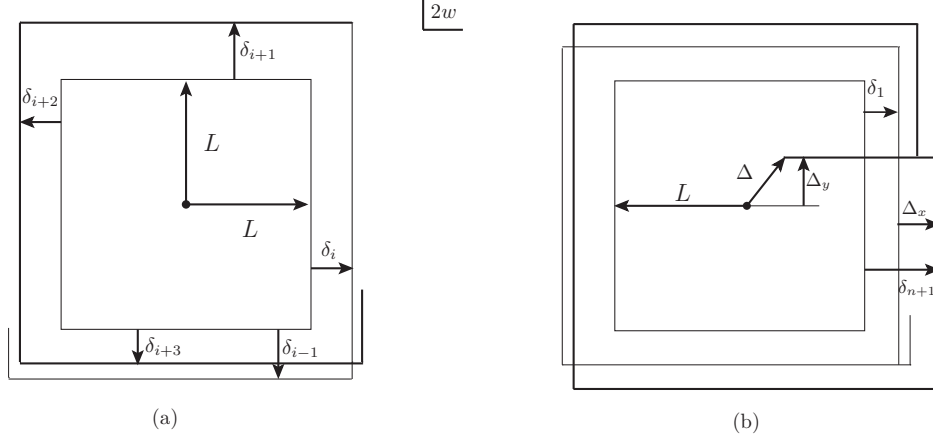


Figure 1: *The cutoff of the surface Σ_0 . Fig (a) shows a portion of the surface in the w -plane. $L = -\log \epsilon_c$ is the cutoff. $\delta_i = -\log \hat{X}_i^+$. The origin should be chosen to be one of zeros of the polynomial $P(z)$. Fig (b) shows that for $n = 4K$ cases the surface is not closed. There is a formal monodromy $\Delta = \Delta_x + i\Delta_y$, thus $\delta_{n+1} = \delta_1 + \Delta_x$, $\delta_{n+2} = \delta_2 + \Delta_y$. The total area is the sum of the area of various rectangles. Notice that we choose to treat the first cusp in a special way. Half of it from δ_1 at the beginning, and half from the end of surface with δ_{n+1} which includes the effect of monodromy.*

Therefore, the cutoff for the radius effectively becomes a cutoff for the w -plane. For example for four consecutive cusps in one cover of w -plane

$$\begin{aligned} \hat{X}_i^+ e^{2\text{Re}[w]} &< \frac{1}{\mu}, & \hat{X}_{i+1}^+ e^{2\text{Im}[w]} &< \frac{1}{\mu}, \\ \hat{X}_{i+2}^+ e^{-2\text{Re}[w]} &> \frac{1}{\mu}, & \hat{X}_{i+3}^+ e^{-2\text{Im}[w]} &> \frac{1}{\mu}, \end{aligned} \quad (18)$$

or equivalently

$$\begin{aligned} 2\text{Re}[w] &< L + \delta_i, & 2\text{Im}[w] &< L + \delta_{i+1}, \\ 2\text{Re}[w] &> -(L + \delta_{i+2}), & 2\text{Im}[w] &> -(L + \delta_{i+3}), \end{aligned} \quad (19)$$

where we have defined

$$\delta_i \equiv -\log \hat{X}_i^+, \quad L \equiv -\log \mu \gg \delta_i. \quad (20)$$

A portion of the regularized surface is shown in Figure 1(a).

Besides using the w coordinate for world-sheet, it is also instrumental to use the spinor representation of $SO(2,4)$ for target space. This was implied firstly from the study in the AdS_3 case [28], where the technique is similar to the spinor helicity formalism (see for example [50, 51]). In AdS_5 case, the spinor representation of $SO(2,4)$ becomes the fundamental of $SU(2,2)$. Very interestingly, this representation is equivalent to that of momentum twistor variables which was first introduced at weak coupling by Hodges in [52] (see also [22]). The smallest solutions s_i of each Stokes sector play exactly the role of momentum twistor variables. And we have the important relations

$$x_{ij}^2 = \frac{\hat{X}_i \cdot \hat{X}_j}{\hat{X}_i^+ \hat{X}_j^+}, \quad \hat{X}_i \cdot \hat{X}_j = \langle s_i s_{i+1} s_j s_{j+1} \rangle, \quad \hat{X}_i^{\alpha\beta} \sim s_i^\alpha \wedge s_{i+1}^\beta. \quad (21)$$

These smallest solutions and their contractions are the basic block of Y system as review in Appendix A.

Notice we can rewrite (21) as

$$\delta_i + \delta_j = \ell_{ij} - \log(\hat{X}_i \cdot \hat{X}_j) , \quad \ell_{ij} \equiv \log x_{ij}^2 . \quad (22)$$

For adjacent case, they are simplified as

$$\delta_i + \delta_{i+2} = \ell_i , \quad \ell_i \equiv \log x_{i,i+2}^2 , \quad (23)$$

where we can use the normalization condition (115), so that $\hat{X}_i \cdot \hat{X}_{i+2} = \langle s_i s_{i+1} s_{i+2} s_{i+3} \rangle = 1$.

3.2 Why $n = 4K$ is special

With the above preparation, we can now calculate the cutoff part. We start from the $n \neq 4K$ case. In such case, the cutoff part is simply given by summing over all rectangles of the surface as shown in Figure 1(a). The whole contribution is

$$A_{\text{cutoff}} = \frac{1}{2} \sum_{i=1}^n (L + \delta_i)(L + \delta_{i+1}) . \quad (24)$$

This can be separated into a divergent part and a finite part as

$$A_{\text{cutoff}} = A_{\text{div}} + A_{\text{BDS-like}} , \quad (25)$$

$$A_{\text{div}} = \frac{1}{2} \sum_{i=1}^n \left(L + \frac{\delta_i + \delta_{i+2}}{2} \right)^2 , \quad (26)$$

$$A_{\text{BDS-like}} = -\frac{1}{4} \sum_{i=1}^n \delta_i (\delta_i + \delta_{i+2} - 2\delta_{i+1}) . \quad (27)$$

Now we need to solve for δ_i in terms of the kinematic variables. For the $n \neq 4K$ case, it is enough to consider the equations involving only adjacent kinematic invariant (23)

$$\delta_i + \delta_{i+2} = \ell_i , \quad i = 1, 2, \dots, n . \quad (28)$$

We also impose the periodic condition $\delta_{i+n} = \delta_i$. Then it is easy to solve these n equations and express δ_i in terms of ℓ_i . By substituting the solution into (27), we obtain exactly the expression of cutoff part (9) and (10).

However, the above prescription is no longer true when $n = 4K$. In particular the periodic condition is no longer allowed. This is because the n equations (23) are decoupled into two sets: one only involves odd indices, the other only involves even indices⁹

$$\begin{aligned} \delta_{2k+1} + \delta_{2k+3} = \ell_{2k+1} &\rightarrow \delta_1 + \delta_3 = \ell_1 , \quad \dots , \quad \delta_{n-1} + \delta_{n+1} = \ell_{n-1} ; \\ \delta_{2k} + \delta_{2k+2} = \ell_{2k} &\rightarrow \delta_2 + \delta_4 = \ell_2 , \quad \dots , \quad \delta_n + \delta_{n+2} = \ell_n . \end{aligned} \quad (29)$$

⁹We emphasize that this degeneracy of equations only appears when $n = 4K$ and is in some sense the root that why it is tricky for such cases.

If we still impose the periodic condition $\delta_{n+i} = \delta_i$, we would have

$$\ell_1 - \ell_3 + \ell_5 - \cdots + \ell_{n-3} - \ell_{n-1} = 0 , \quad (30)$$

$$\ell_2 - \ell_4 + \ell_6 - \cdots + \ell_{n-2} - \ell_n = 0 , \quad (31)$$

which is in general not true. Therefore, we have to break the periodic condition and let

$$\delta_{n+1} = \delta_1 + \Delta_x , \quad \delta_{n+2} = \delta_2 + \Delta_y , \quad (32)$$

by introducing two new variables $\Delta_{x,y}$.

In the w -plane, this non-periodic condition for δ_i means that after going around the w -plane $n/4$ times, the origin of w plane experiences a shift

$$w \rightarrow w + \Delta , \quad \Delta = \Delta_x + i\Delta_y . \quad (33)$$

This is illustrated in Figure 1(b).

This shift can also be understood from another point of view. Notice that world-sheet coordinate transformation is defined as $w = \int P(z)^{1/4} dz$. Since for n points, the degree of polynomial $P(z)$ is $n - 4$, it is *only* in the $n = 4K$ case that there is a single pole for

$$P(z)^{1/4} \sim z^{n/4-1} + \cdots + \frac{c}{z} + \cdots . \quad (34)$$

Therefore the cycle integral is non-zero around infinity in such case. The means that the shift we impose above is actually the monodromy around infinity in the w -plane

$$\Delta \sim \oint_{\gamma^\infty} P(z)^{1/4} dz . \quad (35)$$

By solving the equations (29) and (32), one can express the monodromy in terms of kinematic variable as

$$\Delta_x = -\ell_1 + \ell_3 - \ell_5 + \ell_7 - \cdots + \ell_{n-1} , \quad \Delta_y = -\ell_2 + \ell_4 - \ell_6 + \ell_8 - \cdots + \ell_n . \quad (36)$$

It is interesting to see that this is equivalent to the following relation (by using (21) and (115))

$$\Delta_x = \log \langle s_1 s_2 s_{n-1} s_n \rangle , \quad \Delta_y = \log \langle s_2 s_3 s_n s_{n+1} \rangle , \quad (37)$$

which is related to a T function (113)

$$T_{2,n-4} = \langle s_0 s_1 s_{n-2} s_{n-1} \rangle^{[-n+2]} = \left(e^{-\sqrt{2}(\frac{w_0}{\zeta} + \bar{w}_0 \zeta)} \right)^{[-n+2]} . \quad (38)$$

The second equation was derived in [30], where w_0 is called formal monodromy¹⁰.

$$w_0 = (m_{n-5} + \sqrt{2}m_{n-6} + m_{n-7}) - (m_{n-9} + \sqrt{2}m_{n-10} + m_{n-11}) + \cdots . \quad (39)$$

Via monodromy, this provides one simple relation between mass parameters and kinematic invariants.

¹⁰There is also another parameter μ also contribute to the formal monodromy in [30], which is related to gauge connection, and has no relation with the discussion here.

3.3 Cutoff part of $n=4K$ case

The cutoff surface for the $n=4K$ case has the structure as shown in Figure 1(b). To calculate the cutoff part, we sum over all rectangles as in $n \neq 4K$ cases, but we also need to consider the monodromy contribution. As shown in Figure 1(b), we treat the first cusp in a special way. We separate this cusp into two parts. One part is from δ_1 at the beginning, and the other part from the end of surface with δ_{n+1} , which includes the contribution of monodromy. We choose half of each part so that to have an average contribution. This is similar to the picture used in [28] for AdS_3 case. The whole contribution is

$$A_{\text{cutoff}} = \frac{1}{2} \left[\sum_{i=1}^n (L + \delta_i)(L + \delta_{i+1}) - \frac{1}{2} \Delta_x \Delta_y + (L + \delta_{n+1}) \Delta_y \right]. \quad (40)$$

Notice that we need to take

$$\delta_{n+1} = \delta_1 + \Delta_x, \quad \delta_{n+2} = \delta_2 + \Delta_y, \quad \delta_{n+3} = \delta_3 - \Delta_x, \quad \text{and so on.} \quad (41)$$

Now we need to solve for all δ_i . Due to the monodromy $\Delta_{x,y}$, it is no longer enough to consider only the equations (23). But there are many other equations as given by (22), which involve non-adjacent kinematic invariants

$$\delta_i + \delta_j = \ell_{ij} - \log(\hat{X}_i \cdot \hat{X}_j). \quad (42)$$

To solve our problem, it is enough to choose two of them, for example

$$\delta_1 + \delta_4 = \ell_{14} - \log(\hat{X}_1 \cdot \hat{X}_4), \quad (43)$$

$$\delta_2 + \delta_5 = \ell_{25} - \log(\hat{X}_2 \cdot \hat{X}_5). \quad (44)$$

The price is that we also introduce two new non-trivial variables

$$\hat{X}_1 \cdot \hat{X}_4 = \langle s_1 s_2 s_4 s_5 \rangle, \quad \hat{X}_2 \cdot \hat{X}_5 = \langle s_2 s_3 s_5 s_6 \rangle. \quad (45)$$

This two quantities are related to one of T functions, $T_{2,1} = \langle s_{-2} s_{-1} s_1 s_2 \rangle$. This T functions can be calculated from a limit of Y function as we will show in next section.

Therefore, the cutoff part is finally expressed in terms of kinematic invariants ℓ_{ij} and T functions. The terms that related to the T function will be defined as *extra part*. The remaining parts that only depend on kinematic invariants will be defined as *BDS-like part*. We will provide the explicit expression of eight-point case in section 6.

4 T function as a limit of Y function

In this section, we calculate T functions. We show that the T functions can be obtained as a limit of Y functions. The basic idea is that we can obtain a lower-point structure by taking a limit of a higher-point case. We will first show how to do this in the AdS_3 case. The same calculation is then straightforward to generalize to the AdS_5 case.

It is impossible to review the whole Y system here. However, to make the paper more self-contained, in particular to set up the conventions, we provide a brief summary of Y system in Appendix A. Reader can find more details in [30]

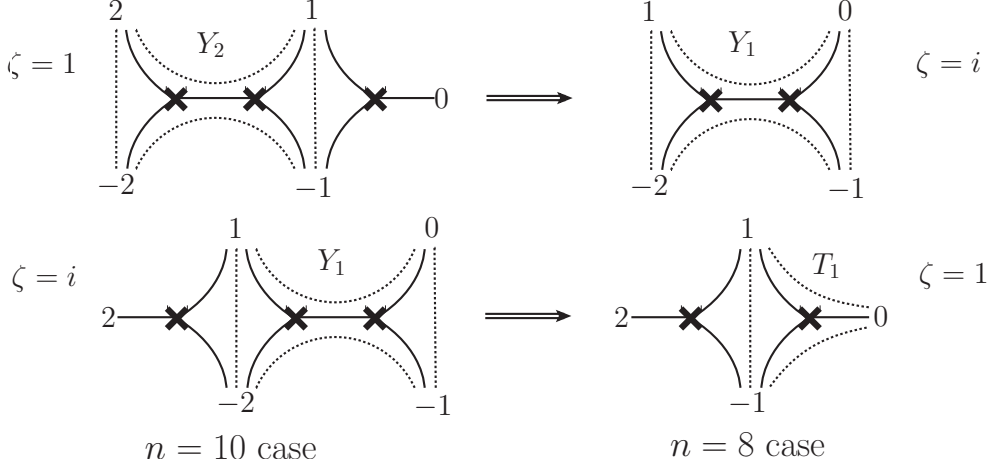


Figure 2: *The limit behavior of the WKB pattern. The crosses are zeros of the polynomial $p(z)$. The numbers indicate the various Stokes sectors. The dotted lines are WKB lines which connect different Stokes sectors. The solid lines ending on the zeros separate different classes of WKB lines. We consider two different phases of ζ , which show the contour formed by WKB lines for Y_2 and Y_1 respectively. By taking the rightmost zero to infinity, the structure of $n=10$ is reduced to that of $n=8$, and Y_2 and Y_1 of the higher-point case are reduced to Y_1 and T_1 of the lower-point case. Notice the change of labels of the Stokes sectors in the limit.*

4.1 The AdS_3 case

We focus on the function T_1 , which will be related to the extra part of the area. We start from two Y functions (see (106))

$$\hat{Y}_1 = Y_1^- = \frac{\langle s_{-2}s_1 \rangle \langle s_{-1}s_0 \rangle}{\langle s_{-2}s_{-1} \rangle \langle s_0s_1 \rangle}, \quad \hat{Y}_2 = Y_2 = \frac{\langle s_{-1}s_1 \rangle \langle s_{-2}s_2 \rangle}{\langle s_{-2}s_{-1} \rangle \langle s_1s_2 \rangle}. \quad (46)$$

The WKB lines corresponding to these two Y function are shown in Figure 2. We consider the limit that the rightmost zero goes to infinity. Notice the WKB lines (dotted lines) that connect different Stokes sectors combine to form a contour which corresponds to a Y or T function as illustrated in the figure. By taking the rightmost zero to infinity, we reduce the n -point structure to the structure of $(n-2)$ -point. We can see explicitly that Y_2 and Y_1 of the higher-point case are reduced to Y_1 and T_1 of a lower-point case. Therefore we have that

$$(\hat{Y}_1^{(n)})^+ \rightarrow \frac{\langle s_{-1}s_1 \rangle \langle s_0s_0 \rangle}{\langle s_{-1}s_0 \rangle \langle s_0s_1 \rangle} \sim \langle s_{-1}s_1 \rangle = T_1^{(n-2)}, \quad (47)$$

$$\hat{Y}_2^{(n)} \rightarrow \left(\frac{\langle s_{-1}s_0 \rangle \langle s_{-2}s_1 \rangle}{\langle s_{-2}s_{-1} \rangle \langle s_0s_1 \rangle} \right)^+ = (\hat{Y}_1^{(n-2)})^+, \quad (48)$$

The superscript of $Y^{(n)}$ means that it is a Y function of the n -point system. The “+” can be obtained by considering the change of the phase of ζ .

One subtly here is that $\hat{Y}_1^{(n)}$ is actually vanishing in this limit, due to the factor $\langle s_0s_0 \rangle$. This is because the limit of taking the zero to infinity actually corresponds to the large m_1 limit, and since for large m_s we have

$$\log Y_s = -m_s \cosh \theta + \dots, \quad (49)$$

Y_1 indeed goes to zero in the large m_1 limit. To evaluate the T_1 function, we renormalize the Y_1 function by subtracting the asymptotic WKB term $Y_{1,\text{WKB}} = e^{-m_s \cosh \theta}$ as

$$T_1^{(n-2)} = \left(\frac{Y_1^{(n)}}{Y_{1,\text{WKB}}^{(n)}} \right) \Big|_{m_1 \rightarrow \infty} . \quad (50)$$

We mention that this is equivalent to making a choice for the normalization of T functions. Unlike Y functions, T functions are not “gauge invariant” due to the gauge redundancy of the Hirota equation [30]. Therefore, such choice of normalization is actually a choice of gauge fixing condition. A more explicit discussion for this point and its relation to periods part can be found in [58].

We can calculate T_1 now. As reviewed in Appendix A, the general integral equations for Y functions are

$$\log Y_s(\theta) = -m_s \cosh \theta + K \star \log(1 + Y_{s+1})(1 + Y_{s-1}) , \quad K(\theta) = \frac{1}{2\pi \cosh(\theta)} . \quad (51)$$

For the first two Y functions we have

$$\log Y_1 = -m_1 \cosh \theta + K \star \log(1 + Y_2) , \quad (52)$$

$$\log Y_2 = -m_2 \cosh \theta + K \star \log(1 + Y_1) + K \star \log(1 + Y_3) . \quad (53)$$

In the large m_1 limit, we have

$$\log T_1^{(n-2)} = \log \left(\frac{Y_1^{(n)}}{Y_{1,\text{WKB}}^{(n)}} \right) \Big|_{m_1 \rightarrow \infty} = K \star \log(1 + Y_2^{(n)\text{lim}}) . \quad (54)$$

We also have

$$\log Y_2^{(n),\text{lim}} = -m_2 \cosh \theta + K \star \log(1 + Y_3^{(n),\text{lim}}) , \quad (55)$$

which is in a integral form of Y_1 , which is consistent with the (48) that $Y_2^{(n)} \rightarrow Y_1^{(n-2)}$ (and more generally $Y_s^{(n)} \rightarrow Y_{s-1}^{(n-2)}$). So we have the T_1 function in the $(n-2)$ -point system as

$$\log T_1 = K \star \log(1 + Y_1) . \quad (56)$$

This may be written in a more explicit form as (for $\varphi_1 \in (-\pi/2, \pi/2)$)

$$\log T_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \frac{1}{\cosh(\theta' - \theta + i\varphi_1)} \log(1 + Y_1(\theta')) . \quad (57)$$

For the case of $n = 8$, this is exactly the same expression of as (6.5) in [28]:

$$\log \gamma_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \frac{1}{\cosh(\theta' - \theta + i\varphi)} \log \left(1 + e^{-|m| \cosh \theta'} \right) , \quad (58)$$

where γ_1 was called “Stokes parameter” there¹¹. This shows that the Stokes parameter can actually be understood as T function. Our definition of the extra part is therefore the same as the definition in [28] which is related to the Stokes parameter.

¹¹Notice one should replace m with $m/2\pi$ for the result in [28] to accord with the convention here.

4.2 The general AdS_5 case

The above prescription can be directly generalized to the AdS_5 case. To calculate the extra part we need to calculate $T_{2,1}$ as shown below (45). We consider $Y_{a,s}$ for $s = 1, 2$ which may be written explicitly as

$$\hat{Y}_{2,1} \equiv Y_{2,1}^- = \frac{\langle -3, -2, 1, 2 \rangle \langle -2, -1, 0, 1 \rangle}{\langle -3, -2, -1, 1 \rangle \langle -2, 0, 1, 2 \rangle}, \quad (59)$$

$$\hat{Y}_{2,2} \equiv Y_{2,2} = \frac{\langle -3, -2, 2, 3 \rangle \langle -2, -1, 1, 2 \rangle}{\langle -3, -2, -1, 2 \rangle \langle -2, 1, 2, 3 \rangle}, \quad (60)$$

$$\hat{Y}_{1,1} \equiv Y_{1,1} = \frac{\langle -1, 0, 1, 2 \rangle \langle -2, 1, 2, 3 \rangle}{\langle 0, 1, 2, 3 \rangle \langle -2, -1, 1, 2 \rangle}, \quad (61)$$

$$\hat{Y}_{1,2} \equiv Y_{1,2}^- = \frac{\langle -4, -3, -2, 2 \rangle \langle -3, -2, -1, 1 \rangle}{\langle -4, -3, -2, -1 \rangle \langle -3, -2, 1, 2 \rangle}, \quad (62)$$

$$\hat{Y}_{3,1} \equiv Y_{3,1} = \frac{\langle -2, -1, 0, 1 \rangle \langle -3, -2, -1, 2 \rangle}{\langle -2, -1, 1, 2 \rangle \langle -3, -2, -1, 0 \rangle}, \quad (63)$$

$$\hat{Y}_{3,2} \equiv Y_{3,2}^- = \frac{\langle -3, 1, 2, 3 \rangle \langle -2, 0, 1, 2 \rangle}{\langle -3, -2, 1, 2 \rangle \langle 0, 1, 2, 3 \rangle}. \quad (64)$$

In exactly the same picture as the AdS_3 case, we take the zero in the rightmost side to infinity and obtain that

$$(\hat{Y}_{2,1}^{(n)})^+ \rightarrow \frac{\langle -2, -1, 1, 2 \rangle \langle -1, 0, 0, 1 \rangle}{\langle -2, -1, 0, 1 \rangle \langle -1, 0, 1, 2 \rangle} \sim \langle -2, -1, 1, 2 \rangle = T_{2,1}^{(n-1)}, \quad (65)$$

$$\hat{Y}_{2,2}^{(n)} \rightarrow \left(\frac{\langle -3, -2, 1, 2 \rangle \langle -2, -1, 0, 1 \rangle}{\langle -3, -2, -1, 1 \rangle \langle -2, 0, 1, 2 \rangle} \right)^+ = (\hat{Y}_{2,1}^{(n-1)})^+ = Y_{2,1}^{(n-1)}, \quad (66)$$

$$\hat{Y}_{1,1}^{(n)} \rightarrow \left(\frac{\langle -1, 0, 0, 1 \rangle \langle -2, 0, 1, 2 \rangle}{\langle 0, 0, 1, 2 \rangle \langle -2, -1, 0, 1 \rangle} \right)^+ \sim (\langle -2, 0, 1, 2 \rangle)^+ = T_{1,1}^{(n-1)}, \quad (67)$$

$$(\hat{Y}_{1,2}^{(n)})^+ \rightarrow \frac{\langle -2, -1, 0, 1 \rangle \langle -3, -2, -1, 2 \rangle}{\langle -2, -1, 1, 2 \rangle \langle -3, -2, -1, 0 \rangle} = Y_{3,1}^{(n-1)}, \quad (68)$$

$$\hat{Y}_{3,1}^{(n)} \rightarrow \left(\frac{\langle -2, -1, 0, 0 \rangle \langle -3, -2, -1, 1 \rangle}{\langle -2, -1, 0, 1 \rangle \langle -3, -2, -1, 0 \rangle} \right)^+ \sim (\langle -3, -2, -1, 1 \rangle)^+ = T_{3,1}^{(n-1)}, \quad (69)$$

$$(\hat{Y}_{3,2}^{(n)})^+ \rightarrow \frac{\langle -1, 0, 1, 2 \rangle \langle -2, 1, 2, 3 \rangle}{\langle 0, 1, 2, 3 \rangle \langle -2, -1, 1, 2 \rangle} = Y_{1,1}^{(n-1)}. \quad (70)$$

There are also some terms go to zero in the limit for $\hat{Y}_{a,1}$. Similar to AdS_3 case, we can renormalize them by subtracting the WKB term as

$$T_{2,1}^{(n-1)} = \left(\frac{Y_{2,1}^{(n)}}{Y_{2,1\text{WKB}}^{(n)}} \right) \Big|_{m_1 \rightarrow \infty}, \quad (71)$$

where $Y_{2,1\text{WKB}} = e^{-\sqrt{2}m_1 \cosh \theta}$. Using the integral form of Y functions (122), we obtain

$$\begin{aligned} \log T_{2,1}^{(n-1)} &= \log \left(\frac{Y_{2,1}^{(n)}}{Y_{2,1\text{WKB}}^{(n)}} \right) \Big|_{m_1 \rightarrow \infty} \\ &= K_2 \star \log(1 + Y_{2,2}^{(n)\text{lim}}) + K_1 \star \log(1 + Y_{1,2}^{(n)\text{lim}})(1 + Y_{3,2}^{(n)\text{lim}}), \end{aligned} \quad (72)$$

From (65)-(70), we know that $Y_{2,2}^{(n)}, Y_{1,2}^{(n)}, Y_{3,2}^{(n)}$ is reduced to $Y_{2,1}^{(n-1)}, Y_{3,1}^{(n-1)}, Y_{1,1}^{(n-1)}$ in the limit, therefore we obtain the $T_{2,1}$ function in a $(n-1)$ -point system as

$$\log T_{2,1} = K_2 \star \log(1 + Y_{2,1}) + K_1 \star \log(1 + Y_{1,1})(1 + Y_{3,1}) . \quad (73)$$

We can also derive the formula for $T_{1,1}$ and $T_{3,1}$ in the same way. The finally expressions are

$$\log T_{1,1} = \frac{1}{2} K_2 \star \log(1 + Y_{1,1})(1 + Y_{3,1}) + K_1 \star \log(1 + Y_{2,1}) - \frac{1}{2} K_3 \star \log \frac{(1 + Y_{1,1})}{(1 + Y_{3,1})} , \quad (74)$$

$$\log T_{3,1} = \frac{1}{2} K_2 \star \log(1 + Y_{1,1})(1 + Y_{3,1}) + K_1 \star \log(1 + Y_{2,1}) + \frac{1}{2} K_3 \star \log \frac{(1 + Y_{1,1})}{(1 + Y_{3,1})} . \quad (75)$$

It is easy to check these results indeed yield the required functional relations (similar to what was done for Y function in [30])

$$T_{1,1}^+ T_{3,1}^- = T_{2,1}(1 + Y_{3,1}) , \quad T_{3,1}^+ T_{1,1}^- = T_{2,1}(1 + Y_{1,1}) , \quad T_{2,1}^+ T_{2,1}^- = T_{1,1} T_{3,1}(1 + Y_{2,1}) , \quad (76)$$

by using the identities of kernels that

$$K_2^+ + K_2^- = \delta(\theta) + 2K_1 , \quad K_1^+ + K_1^- = K_2 , \quad K_3^+ - K_3^- = -\delta(\theta) . \quad (77)$$

In the above integral form, the T functions can be calculated in the same way as Y functions.

5 A conjecture for periods part

The final missing piece is the periods part. As we mentioned in section 2, the periods part is the difference between the surface Σ and simplified surface Σ_0 . It contains the branch cut information which depends on the polynomial $P(z)$. It also depends on how we choose the surface Σ_0 . For the $n \neq 4K$ case, the Σ_0 surface can be defined by choosing a simple polynomial $P_0(z) = z^{n-4}$. Periods part can then be given explicitly as

$$A_{\text{periods}}^{n \neq 4K} = 2 \int d^2 z ([P(z)\bar{P}(\bar{z})]^{1/4} - |z|^{n/2-2}) . \quad (78)$$

For the case that $n = 4K$, due to the monodromy, we cannot choose such a simple polynomial for Σ_0 .

As we mentioned before, the periods part is expressed in terms of periods around cycles of the Riemann surface. The corresponding Riemann surface for AdS_5 case is defined as $x^4 = P(z)$ which is a quadruple cover of Riemann sphere. While for AdS_3 case, it is a simpler double branch cover given as $x^2 = p(z)$, and the periods part was given in [28]. We will first review the result of AdS_3 case, and then make a direct generalization to AdS_5 case.

To study the periods part, we need to choose a basis of cycle for the Riemann surface. Following [28], we choose

$$n = 4K + 2 : \quad \gamma^s , \quad s = 1, \dots, \frac{n-6}{2} , \quad (79)$$

$$n = 4K : \quad \gamma^s , \quad s = 2, \dots, \frac{n-8}{2} , \quad \text{and } \gamma^\infty , \gamma_m^\infty , \quad (80)$$

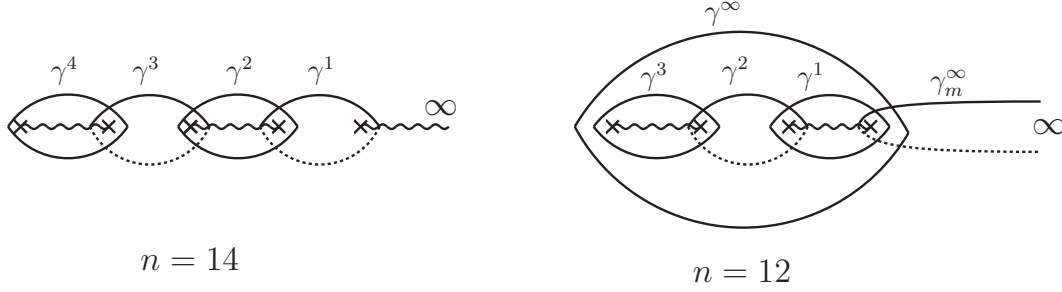


Figure 3: The pattern of cycle structure for the Riemann surface. The crosses represent the zeros of polynomial $p(z)$. The wave lines indicate the branch cuts. There is a branch point at infinity for $n = 4K + 2$ case. Notice the the cycle γ_m^∞ in $n = 12$ may be taken as the the cycle γ^1 in $n = 14$ by taking the rightmost zero to infinity.

where the case of $n = 14$ and $n = 12$ are shown explicitly in Figure 3. Other cases have similar patterns.

We can see that while all cycles are compact for $n = 4K + 2$ case, there is a non-compact cycle γ_m^∞ when $n = 4K$. Its dual cycle γ^∞ goes around infinity, over which the integration gives the monodromy we discussed before. As mentioned in [28], to obtain the correct normalization of Stokes parameter (i.e. the T function), we need to choose the origin of w -plane to be the zero which the non-compact cycle is around. The results of periods part are given in [28], which involves only non-infinite cycles. The final expression of the periods part turns out to have the same expression as $n = 4K - 2$ case¹². The periods part can therefore be explicit defined as

$$A_{\text{periods}}^{n=4K} = 2 \int d^2 z \left([\tilde{p}(z)\tilde{\bar{p}}(\bar{z})]^{1/2} - |z|^{\frac{n-2}{2}-2} \right), \quad \tilde{p}(z) = \frac{p(z)}{z - z_1}, \quad (81)$$

where z_1 is the zero in the most right side, which both γ^∞ and γ_m^∞ go around, and is defined as the origin of the Σ_0 surface. For the $n = 8$ case in AdS_3 , the periods part is same as six-point case and therefore should be zero. This is indeed true, for example let $p(z) = z^2 - a^2$ and $\tilde{p}(z) = z + a$, we have

$$A_{\text{periods}}^{n=8, AdS_3} = 2 \int d^2 z \left(\sqrt{(z+a)(\bar{z}+\bar{a})} - \sqrt{z\bar{z}} \right) = 0. \quad (82)$$

Following the above picture, we may conjecture the periods part in AdS_5 case to be similarly given as

$$A_{\text{periods}}^{n=4K, \text{conjecture}} = 2 \int d^2 z \left([\tilde{P}(z)\tilde{\bar{P}}(\bar{z})]^{1/4} - |z|^{\frac{n-2}{2}-2} \right), \quad \tilde{P}(z) = \frac{P(z)}{z - z_1}, \quad (83)$$

where under this assumption, the $4K$ -point result should have the same expression as the $(4K-1)$ -point case by replacing m_s with m_{s+1} , for example for eight-point we may obtain from the result of seven points as

$$A_{\text{periods}}^{n=8, \text{conjecture}} = \frac{|m_2|^2 + |m_3|^2}{2} + \frac{m_2 \bar{m}_3 + \bar{m}_2 m_3}{2\sqrt{2}}. \quad (84)$$

¹²One may understand this point by considering that the information of the two infinite cycles, which is related to the monodromy, is already included in the cutoff part.

Since the structure of Riemann surfaces is more complicated than AdS_3 case, it may not be surprising if there are extra contribution in AdS_5 case. It would be important to check whether this generalization is correct or not¹³.

6 Eight-point result

In this section we present the explicit calculation for eight-point case. The cutoff part is given as

$$A_{\text{cutoff}} = A_{\text{div}} + A_{\text{fincut}} , \quad (85)$$

$$A_{\text{div}} = \frac{1}{2} \sum_{i=1}^8 \left(L + \frac{\delta_i + \delta_{i+2}}{2} \right)^2 , \quad (86)$$

$$A_{\text{fincut}} = -\frac{1}{8} \sum_{i=1}^8 \ell_i^2 + \frac{1}{2} \sum_{i=1}^n \delta_i \delta_{i+1} + \frac{1}{4} \Delta_x \Delta_y + \frac{1}{2} \delta_1 \Delta_y . \quad (87)$$

where A_{fincut} contains both BDS-like and extra part. Notice $\delta_9 = \delta_1 + \Delta_x$, $\delta_{10} = \delta_2 + \Delta_y$. We can solve δ_i plus $\Delta_{x,y}$ from equations (23), (43) and (44), which we collected here

$$\delta_i + \delta_{i+2} = \ell_i \equiv \log x_{i,i+2}^2 , \quad (88)$$

$$\delta_1 + \delta_4 = \ell_{14} - \log T_{2,1}^{[6]} , \quad (89)$$

$$\delta_2 + \delta_5 = \ell_{25} - \log T_{2,1}^{[8]} , \quad (90)$$

where we have used the relation $T_{2,1}^{[6]} = \langle s_1 s_2 s_4 s_5 \rangle$ and $T_{2,1}^{[8]} = \langle s_2 s_3 s_5 s_6 \rangle$. We write the monodromy explicitly

$$\Delta_x = -\ell_1 + \ell_3 - \ell_5 + \ell_7 = \log \langle s_1 s_2 s_7 s_8 \rangle , \quad \Delta_y = -\ell_2 + \ell_4 - \ell_6 + \ell_8 = \log \langle s_2 s_3 s_8 s_9 \rangle . \quad (91)$$

where we have used (37).

The parts related to T functions are define as extra part

$$A_{\text{extra}} = -\frac{1}{4} \left[(\Delta_x + \Delta_y) \log T_{2,1}^{[6]} - (\Delta_x - \Delta_y) \log T_{2,1}^{[8]} \right] , \quad (92)$$

where $T_{2,1}$ function can be calculated using (73). Notice that to calculate $T_{2,1}^{[6,8]}$, one needs to generalize the equation (73) to other phase regions of ζ -plane, where pole terms should be included [30]. The BDS-like part is given by the remaining parts as

$$\begin{aligned} A_{\text{BDS-like}} &= A_{\text{fincut}} - A_{\text{extra}} \\ &= \frac{1}{8} \sum_{i=1}^8 [\ell_i^2 - (\ell_i - \ell_{i+1})^2] + \frac{1}{4} [\ell_1 \Delta_x + \ell_2 \Delta_y - \ell_3 \Delta_x - \ell_4 (\ell_3 + \ell_7) - \ell_8 (\ell_1 + \ell_5)] \\ &\quad + \frac{1}{4} [(\Delta_x + \Delta_y) \ell_{14} - (\Delta_x - \Delta_y) \ell_{25}] . \end{aligned} \quad (93)$$

¹³Note added: The periods parts are calculated in a later paper by the author by considering a special collinear limit [58]. In this limit, the periods part can be uniquely fixed by the BDS part (the one-loop finite part of amplitudes at weak coupling[1]), and the cutoff part calculated in this paper. There is indeed extra contribution compared to the conjecture here. We cite the correct periods part of eight-point in next section (94).

Other parts of the amplitude are¹⁴

$$A_{\text{periods}} = \frac{|m_2|^2 + |m_3|^2}{2} + \frac{m_2 \bar{m}_3 + \bar{m}_2 m_3}{2\sqrt{2}} - \frac{1}{4}|m_1 + \sqrt{2}m_2 + m_3|^2, \quad (94)$$

$$A_{\text{free}} = \sum_{s=1}^3 \frac{|m_s|}{2\pi} \int_{-\infty}^{+\infty} d\theta \cosh \theta \log \left[(1 + Y_{1,s})(1 + Y_{3,s})(1 + Y_{2,s})^{\sqrt{2}} \right]. \quad (95)$$

The whole area is (up to a constant)

$$A = A_{\text{div}} + A_{\text{BDS-like}} + A_{\text{extra}} + A_{\text{periods}} + A_{\text{free}}. \quad (96)$$

6.1 Another choice of equations

To calculate δ_i , we have chosen two extra conditions (43) and (44). We may choose other equations as well. For example

$$\delta_8 + \delta_4 = \log x_{48}^2 - \log(\hat{X}_8^+ \cdot \hat{X}_4^+) = \ell_{48} - \log T_{2,2}^{[5]}, \quad (97)$$

$$\delta_7 + \delta_3 = \log x_{37}^2 - \log(\hat{X}_7^+ \cdot \hat{X}_3^+) = \ell_{37} - \log T_{2,2}^{[3]}. \quad (98)$$

We obtain a different expression for BDS-like and extra part

$$A_{\text{BDS-like}} = -\frac{1}{8} \sum_{i=1}^8 \ell_i^2 + \frac{1}{4} \sum_{i=1}^8 \ell_i \ell_{i+1} - \frac{1}{4}(\ell_2 + \ell_6)(\ell_3 + \ell_7) + \frac{1}{4}(\Delta_x \ell_{48} - \Delta_y \ell_{37}), \quad (99)$$

$$A_{\text{extra}} = -\frac{1}{4} \left(\Delta_x \log T_{2,2}^{[-3]} - \Delta_y \log T_{2,2}^{[-5]} \right), \quad (100)$$

With such choices we have to evaluate the $T_{2,2}$ function $\langle s_i s_{i+1} s_{i+4} s_{i+5} \rangle$, which can be calculated by using the relation $T_{2,2} = T_{1,1} T_{3,1} Y_{2,1}$.

We can see that expression of BDS-like and extra parts depends on the choice of equations. There is no unique definition for each of them. However, the summation of extra and BDS-like part must be invariant. We can check this explicit.

The difference between the BDS-like part is

$$\begin{aligned} (93) - (99) &= \frac{1}{4}(\ell_1 + \ell_5 - \ell_3 - \ell_7)(\ell_1 + \ell_{2,5} + \ell_{4,8} - \ell_3 - \ell_8 - \ell_{1,4}) \\ &\quad + \frac{1}{4}(\ell_2 + \ell_6 - \ell_4 - \ell_8)(\ell_2 + \ell_3 + \ell_7 + \ell_{1,4} + \ell_{2,5} - \ell_{3,7}) \\ &= -\frac{1}{4} \Delta_x \log \left(\frac{x_{13}^2 x_{25}^2 x_{48}^2}{x_{35}^2 x_{82}^2 x_{14}^2} \right) - \frac{1}{4} \Delta_y \log \left(\frac{x_{24}^2 x_{35}^2 x_{71}^2}{x_{14}^2 x_{25}^2 x_{37}^2} \right) \\ &= -\frac{1}{4} \Delta_x \log \left(\frac{\langle s_2 s_3 s_5 s_6 \rangle \langle s_4 s_5 s_8 s_9 \rangle}{\langle s_1 s_2 s_4 s_5 \rangle} \right) + \frac{1}{4} \Delta_y \log (\langle s_1 s_2 s_4 s_5 \rangle \langle s_2 s_3 s_5 s_6 \rangle \langle s_3 s_4 s_7 s_8 \rangle), \end{aligned}$$

This exactly cancels the difference between the extra parts (92) – (100), by noticing that

$$T_{2,1}^{[6]} = \langle s_1 s_2 s_4 s_5 \rangle, \quad T_{2,1}^{[8]} = \langle s_2 s_3 s_5 s_6 \rangle, \quad T_{2,2}^{[-3]} = \langle s_4 s_5 s_8 s_9 \rangle, \quad T_{2,2}^{[-5]} = \langle s_3 s_4 s_7 s_8 \rangle. \quad (101)$$

Therefore, the cutoff parts with two different choices of equations are indeed equivalent to each other.

¹⁴We have cited the results of periods part obtained in [58], with which the amplitude has the correct collinear limit.

7 Discussion

Let us make more comments on the BDS-like part. First notice that the BDS-like part (plus the universal divergent part) gives the correct dual conformal anomaly, since all other parts are conformal invariant. Therefore the difference between BDS-like and BDS part must be a conformal invariant functions. As we have seen that in $n=4K$ case, we do not have a unique definition of BDS-like part. This is different from the $n \neq 4K$ case, in which the BDS-like part is uniquely expressed in terms of only adjacent kinematic invariants. The reason for this uniqueness is that, in $n \neq 4K$ case, we cannot express any cross ratios in terms of only adjacent kinematic invariants, therefore the expression is fixed by conformal Ward identity.

We may explain this point more explicitly. Suppose we can express the BDS-like part in terms of only adjacent kinematic invariants ℓ_i , i.e. $A_{\text{BDS-like}} = F(\ell_i)$, which gives correct conformal anomaly. The expression will not be unique if we can also express some function of cross ratios u in terms of only ℓ_i , for example $g(u) = f(\ell_i)$. This is because we can define a new function $A'_{\text{BDS-like}} = F(\ell_i) + f(\ell_i)$ which also satisfies the Ward identity. In the $n \neq 4K$ case, we cannot have any relation as $g(u) = f(\ell_i)$, therefore the function $F(\ell_i)$ is uniquely fixed by Ward identity. But when $n = 4K$, we do have such relations as $g(u) = f(\ell_i)$. At the same time, it is also impossible to have a function $F(\ell_i)$ which can give correct anomaly. Therefore, non-adjacent kinematic invariants are necessary to appear in the final expression. We have seen this explicitly, since we must introduce two new equations for calculating δ_i , which involve non-adjacent kinematic invariants.

Besides the choice of equations, we also made several other choices during the calculation. When computing the cutoff area, we treated the first cusp in a special way. We make some gauge choice which is related to the normalization of T functions. While considering the periods part, we choose the origin of w -plane to be one of the zeros of the polynomial, which is also implicitly related to the normalization of T functions. Of course, the physics i.e. the whole result should be independent of all these choices, as we have checked for the cutoff part. We emphasize that we only make a conjecture for the periods part in this paper, and it would be important to calculate this part more honestly and check the conjecture.

Finally, we mention that there are other important open problems, of which the most challenging one is perhaps how to calculate the amplitude at arbitrary value of 't Hooft coupling constant. One can expect the quantum integrability [53, 54] should play an essential role to realize this. It would also be interesting to study and see if we can apply these method to study the S -matrix in a cousin of $\mathcal{N} = 4$ SYM, the ABJM theory [55]. Some observations for amplitude and Wilson loop duality at weak coupling side are given in [56, 57].

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A A brief summary of Y system

Here we summarize the main result of Y system, and at the same time set up the convention. Reader can find details in [30].

A.1 AdS_3 case

We use the convention that (the convention will be different in AdS_5 case)

$$f^\pm(\zeta) = f(e^{\pm i\pi/2}\zeta) , \quad f^{[m]}(\zeta) = f(e^{im\pi/2}\zeta) . \quad (102)$$

T functions are defined as

$$T_{1,2k+1} = \langle s_{-k-1}s_{k+1} \rangle , \quad T_{1,2k} = \langle s_{-k-1}s_k \rangle^+ , \quad (103)$$

$$T_{0,2k} = \langle s_{-k-1}s_{-k} \rangle , \quad T_{0,2k+1} = \langle s_{-k-2}s_{-k-1} \rangle^+ , \quad (104)$$

$$T_{2,2k} = \langle s_k s_{k+1} \rangle , \quad T_{2,2k+1} = \langle s_k s_{k+1} \rangle^+ , \quad (105)$$

where the contraction of smallest solution is defined as $\langle s_i s_j \rangle = \epsilon_{\alpha\beta} s_i^\alpha s_j^\beta$.

Y functions are defined as

$$Y_m = \frac{T_{1,m-1}T_{1,m+1}}{T_{0,m}T_{2,m}} . \quad (106)$$

We can choose the normalization conditions

$$\langle s_i s_{i+1} \rangle = 1 , \quad (107)$$

which is the gauge fixing conditions for T functions. We also have the shifting relation

$$\langle s_i s_j \rangle^{[2]} = \langle s_{i+1} s_{j+1} \rangle , \quad (108)$$

which comes from the Z_2 symmetry of the corresponding $SU(2)$ Hitchin system.

The Y functions satisfy the functional relations

$$Y_s^- Y_s^+ = (1 + Y_{s-1})(1 + Y_{s+1}) , \quad s = 1, 2, \dots, \frac{n}{2} - 3 . \quad (109)$$

The equivalent integral form can be given as

$$\log Y_s(\theta) = -m_s \cosh \theta + K \star \log(1 + Y_{s+1})(1 + Y_{s-1}) , \quad K(\theta) = \frac{1}{2\pi \cosh \theta} . \quad (110)$$

Notice that in this form we assume the phase φ_s of m_s to be zero, and the valid range for the phase of ζ (or the imaginary part of θ) is $\phi \in (-\pi/2, \pi/2)$.

A.2 AdS_5 case

We use a different convention that

$$f^\pm(\zeta) = f(e^{\pm i\pi/4}\zeta) , \quad f^{[m]}(\zeta) = f(e^{im\pi/4}\zeta) . \quad (111)$$

T functions are defined as

$$T_{0,m} = \langle s_m s_{m+1} s_{m+2} s_{m+3} \rangle^{[-m-1]}, \quad T_{4,m} = \langle s_{-2} s_{-1} s_0 s_1 \rangle^{[-m-1]}, \quad (112)$$

$$T_{1,m} = \langle s_{-2} s_{-1} s_0 s_{m+1} \rangle^{[-m]}, \quad T_{2,m} = \langle s_{-1} s_0 s_{m+1} s_{m+2} \rangle^{[-m-1]}, \quad T_{3,m} = \langle s_{-1} s_m s_{m+1} s_{m+2} \rangle^{[-m]}. \quad (113)$$

where the contraction is defined as $\langle s_i s_{i+1} s_j s_{j+1} \rangle \equiv \epsilon^{\alpha\beta\gamma\delta} s_{i,\alpha} s_{i+1,\beta} s_{j,\gamma} s_{j+1,\delta}$.

Y functions are defined as

$$Y_{a,m} = \frac{T_{a,m-1} T_{a,m+1}}{T_{a-1,m} T_{a+1,m}}. \quad (114)$$

We have the normalization condition that

$$\langle s_i s_{i+1} s_{i+2} s_{i+3} \rangle = 1, \quad (115)$$

and the Z_4 symmetry provides the shifting relations

$$\langle s_{j-1} s_j s_{k-1} s_k \rangle^{[2]} = \langle s_j s_{j+1} s_k s_{k+1} \rangle, \quad (116)$$

$$\langle s_{j-2} s_{j-1} s_j s_k \rangle^{[2]} = \langle s_j s_k s_{k+1} s_{k+2} \rangle, \quad (117)$$

$$\langle s_j s_{k-2} s_{k-1} s_k \rangle^{[2]} = \langle s_j s_{j+1} s_{j+2} s_k \rangle. \quad (118)$$

The Y functions satisfy the following functional relations

$$\frac{Y_{2,s}^- Y_{2,s}^+}{Y_{1,s} Y_{3,s}} = \frac{(1 + Y_{2,s+1})(1 + Y_{2,s-1})}{(1 + Y_{1,s})(1 + Y_{3,s})}, \quad (119)$$

$$\frac{Y_{3,s}^- Y_{1,s}^+}{Y_{2,s}} = \frac{(1 + Y_{3,s+1})(1 + Y_{1,s-1})}{1 + Y_{2,s}}, \quad (120)$$

$$\frac{Y_{1,s}^- Y_{3,s}^+}{Y_{2,s}} = \frac{(1 + Y_{1,s+1})(1 + Y_{3,s-1})}{1 + Y_{2,s}}, \quad (121)$$

where for n -point, $s = 1, 2, \dots, n-5$. The equivalent integral form is

$$\log Y_{2,s} = -\sqrt{2} m_s \cosh \theta - K_2 \star \alpha_s - K_1 \star \beta_s, \quad (122)$$

$$\log Y_{1,s} = -m_s \cosh \theta - C_s - \frac{1}{2} K_2 \star \beta_s - K_1 \star \alpha_s - \frac{1}{2} K_3 \star \gamma_s, \quad (123)$$

$$\log Y_{3,s} = -m_s \cosh \theta + C_s - \frac{1}{2} K_2 \star \beta_s - K_1 \star \alpha_s + \frac{1}{2} K_3 \star \gamma_s, \quad (124)$$

where

$$\alpha_s \equiv \log \frac{(1 + Y_{1,s})(1 + Y_{3,s})}{(1 + Y_{2,s-1})(1 + Y_{2,s+1})}, \quad \gamma_s \equiv \log \frac{(1 + Y_{1,s-1})(1 + Y_{3,s+1})}{(1 + Y_{1,s+1})(1 + Y_{3,s-1})}, \quad (125)$$

$$\beta_s \equiv \log \frac{(1 + Y_{2,s})^2}{(1 + Y_{1,s-1})(1 + Y_{1,s+1})(1 + Y_{3,s-1})(1 + Y_{3,s+1})}, \quad (126)$$

and

$$K_1 \equiv \frac{1}{2\pi} \frac{1}{\cosh \theta}, \quad K_2 \equiv \frac{\sqrt{2}}{\pi} \frac{\cosh \theta}{\cosh 2\theta}, \quad K_3 \equiv \frac{i}{\pi} \tanh 2\theta. \quad (127)$$

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